

ROM2F/2001/01

IHP-2000/10

hep-th/0101074

All Couplings of Minimal Six-dimensional Supergravity

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Abstract

We describe the complete coupling of $(1, 0)$ six-dimensional supergravity to tensor, vector and hypermultiplets. The generalized Green-Schwarz mechanism implies that the resulting theory embodies factorized gauge and supersymmetry anomalies, to be disposed of by fermion loops. Consequently, the low-energy theory is determined by the Wess-Zumino consistency conditions, rather than by the requirement of supersymmetry. As already shown for the case without hypermultiplets, this procedure does not fix a quartic coupling for the gauginos. With respect to these previous results, the inclusion of charged hypermultiplets gives additional terms in the supersymmetry anomaly. We also consider the case in which abelian vectors are present. As in the absence of hypermultiplets, abelian vectors allow additional couplings. Finally, we apply the Pasti-Sorokin-Tonin prescription to this model.

(January , 2001)

1 Introduction

Perturbative six-dimensional string vacua with minimal supersymmetry can arise for instance as compactifications of the heterotic string on $K3$, or as parameter-space orbifolds (orientifolds) [1] of $K3$ reductions of the type-IIB string. While in the former case only a single tensor multiplet is present, in the latter one obtains vacua with variable numbers of tensor multiplets [2], related by string dualities to non-perturbative heterotic and M-theory vacua. In these models, the anomalous contribution due to fermion loops is derived from the residual anomaly polynomial

$$c_x^r c_y^s \eta_{rs} \operatorname{tr}_x F^2 \operatorname{tr}_y F^2 \quad ,$$

where the c 's are a collection of constants (x and y run over the various semi-simple Lie factors in the gauge group and over the Lorentz group) and η is the Minkowski metric for $SO(1, n_T)$, with n_T the number of tensor multiplets [3]. As a consequence, several antisymmetric tensors take part in a generalized Green-Schwarz mechanism [4, 3]. The corresponding Green-Schwarz term has the form

$$B^r c_r^x \operatorname{tr}_x F^2$$

and, if one considers only gauge anomalies, contains only two derivatives, and thus belongs to the low-energy effective action. Consequently, the resulting low-energy lagrangian has a “classical” gauge anomaly, that the Wess-Zumino conditions [5] relate to a “classical” supersymmetry anomaly [6].

The complete coupling of (1,0) six-dimensional supergravity to non-abelian vector and tensor multiplets, obtained in [7] requiring the closure of the Wess-Zumino conditions, has revealed another related aspect of these six-dimensional models: a quartic coupling for the gauginos is undetermined, and the construction is consistent for any choice of this coupling. Correspondingly, the commutator of two supersymmetry transformations on the gauginos contains an extension, that plays a crucial role in ensuring that the Wess-Zumino consistency conditions close on-shell. The coupling of (1,0) six-dimensional supergravity to non-abelian vectors and self-dual tensors reveals neatly the realization of a peculiar

aspect of the physics of branes: singularities in the gauge couplings appear for particular values of the scalars in the tensor multiplets [3], and can be ascribed to a phase transition [8] in which a string becomes tensionless [9]. Moreover, as was shown in [10], in this model the divergence of the energy-momentum tensor is non-vanishing, as is properly the case for a theory that has gauge anomalies but no gravitational anomalies (gravitational anomalies could be accounted for introducing higher-derivative couplings). The whole construction can also be repeated with the inclusion of abelian vectors, that actually allow more general couplings, since in this case the residual anomaly polynomial can have the more general form

$$c_{ab}^r c_{cd}^s \eta_{rs} F^a \wedge F^b \wedge F^c \wedge F^d \quad ,$$

where the indices a, b, c, d run over the different $U(1)$ gauge groups, and where the c^r 's are symmetric matrices that may not be simultaneously diagonalized [11].

Notice that these low-energy couplings are obtained by consistency once one includes the Green-Schwarz term in the low-energy theory. The complete theory, supersymmetric and gauge-invariant, would also include additional non-local couplings arising from fermion loops. This is exactly as in the ten-dimensional case, what is peculiar of these six-dimensional models is that here the anomalous terms belong to the low-energy effective action.

In order to have an explicit realization of the low-energy dynamics of six-dimensional string vacua, it is of interest to consider how the whole construction is modified by the inclusion of hypermultiplets. In [12], the complete coupling to a single tensor multiplet and to vector and charged hypermultiplets was obtained for the case in which no anomalies and no singular couplings are present. More recently, an analysis of the case in which various tensor multiplets are present was carried out in [13], however without taking into account the anomalous terms. Still, this analysis shows that, in correspondence to the phase transition, additional singular terms appear because of the presence of charged hypermultiplets.

In this paper we construct the complete coupling of (1,0) supergravity to all possible (1,0) multiplets, generalizing the results of [7] in order to include hypermultiplets, and

extending the results of [13] to all orders in the fermi fields, while taking into account the anomalous couplings. We show that the inclusion of charged hypermultiplets gives additional terms in the supersymmetry anomaly. As was the case without hypermultiplets [7], the resulting theory is determined up to a quartic coupling for the gauginos, and correspondingly the supersymmetry algebra contains an extension that guarantees the consistency of the construction. Following [12, 13], we will consider the case in which the scalars in the hypermultiplets parametrize the coset $USp(2, 2n_H)/USp(2) \times USp(2n_H)$, and we will describe the gauging of the full compact subgroup $USp(2) \times USp(2n_H)$ of the isometry group $USp(2, 2n_H)$. Other cases, in which the scalars parametrize more general quaternionic symmetric spaces or are charged with respect to different subgroups of the isometry group, can be straightforwardly obtained from our results.

The paper is organized as follows. In section 2 we construct the complete (1,0) supergravity coupled to n_T tensor multiplets, non-abelian vector multiplets and n_H hypermultiplets. In section 3 we describe the case in which abelian vectors are included. Section 4 is devoted to a discussion, in which we also show how to apply the Pasti-Sorokin-Tonin (PST) construction [14] to this model. Finally, the appendix collects some details on the notation and some useful identities.

2 Supersymmetry algebra and equations of motion

In this section we describe the full coupling of six-dimensional supergravity to vector, tensor and hypermultiplets. We will use notations similar to the ones of [7] for what concerns the coupling to vector and tensor multiplets, while in the description of the coupling to hypermultiplets we will follow the notation of [12]. Some details about our conventions are contained in the appendix.

We first summarize the field content of the theory. The gravitational multiplet contains the vielbein e_μ^m , a 2-form and a left-handed gravitino ψ_μ^A , the tensor multiplet contains a 2-form, a scalar and a right-handed tensorino, the vector multiplet contains a vector A_μ and a left-handed gaugino λ^A , and finally the hypermultiplet contains four scalars

and a right-handed hyperino. In the presence of n_T tensor multiplets, the tensorinos are denoted by χ^{MA} where $M = 1, \dots, n_T$ is an $SO(n_T)$ index. The index $A = 1, 2$ is in the fundamental representation of $USp(2)$, and the gravitino, the tensorinos and the gauginos are $USp(2)$ doublets satisfying the symplectic-Majorana condition

$$\psi^A = \epsilon^{AB} C \bar{\psi}_B^T \quad . \quad (2.1)$$

The n_T scalars in the tensor multiplets parametrize the coset $SO(1, n_T)/SO(n_T)$, while the $(n_T + 1)$ 2-forms from the gravitational and tensor multiplets are collectively denoted by $B_{\mu\nu}^r$, with $r = 0, \dots, n_T$ in the fundamental representation of $SO(1, n_T)$, and their field-strengths satisfy (anti)self-duality conditions. The vector and the gaugino are in the adjoint representation of the gauge group. Finally, taking into account n_H hypermultiplets, the hyperinos are denoted by Ψ^a , where $a = 1, \dots, 2n_H$ is a $USp(2n_H)$ index, and the symplectic-Majorana condition for these spinors is

$$\Psi^a = \Omega^{ab} C \bar{\Psi}_b^T \quad , \quad (2.2)$$

where Ω^{ab} is the antisymmetric invariant tensor of $USp(2n_H)$ (see the appendix for more details). The hyper-scalars ϕ^α , $\alpha = 1, \dots, 4n_H$, are coordinates of a quaternionic manifold, that is a manifold whose holonomy group is contained in $USp(2) \times USp(2n_H)$.

If the quaternionic manifold parametrized by the hyper-scalars has isometries, these correspond to global symmetries of the supergravity theory. Then the global symmetry group, or a subgroup thereof, can be gauged. Following [12, 13], we will consider without loss of generality the case in which the scalars parametrize the symmetric manifold $USp(2, 2n_H)/USp(2) \times USp(2n_H)$, whose isometry group is $USp(2, 2n_H)$. We will then describe the gauging of the maximal compact subgroup $USp(2) \times USp(2n_H)$ of the isometry group. All the results can be naturally generalized to other symmetric quaternionic spaces ¹.

The scalars in the tensor multiplets can be described, following [15], in terms of the $SO(1, n_T)$ matrix

$$V = \begin{pmatrix} v_r \\ x_r^M \end{pmatrix} \quad , \quad (2.3)$$

¹I am grateful to S. Ferrara for discussions about this point.

whose elements satisfy the constraints

$$v^r v_r = 1 \quad , \quad v_r v_s - x_r^M x_s^M = \eta_{rs} \quad , \quad v^r x_r^M = 0 \quad . \quad (2.4)$$

In the following, we will take v_r and x_r^M , with the constraints of eq. (2.4), as fundamental fields, as in [15], so that the composite $SO(n_T)$ connection that appears in the covariant derivative of the tensorinos will be $x_r^N \partial_\mu x^{Mr}$. On the other hand, the notation of [13], in which the fundamental fields are the scalars $\Phi^{\bar{\alpha}}$ ($\bar{\alpha} = 1, \dots, n_T$) parametrizing the coset manifold, adds to the supersymmetry variation of the tensorinos χ^{MA} the term

$$- \delta \Phi^{\bar{\alpha}} \mathcal{A}_{\bar{\alpha}}^{MN} \chi^{NA} \quad , \quad (2.5)$$

where $\mathcal{A}_{\bar{\alpha}}^{MN}$ is the composite connection of $SO(n_T)$ [13]. In this notation, the commutator of two supersymmetry transformations on the tensorinos does not generate a local $SO(n_T)$ transformation.

We now recall the notations used to describe the scalars in the hypermultiplets. We denote by $V_\alpha^{aA}(\phi)$ the vielbein of the quaternionic manifold, where the index structure corresponds to the requirement that the holonomy be contained in $USp(2) \times USp(2n_H)$. The internal $USp(2)$ and $USp(2n_H)$ connections are then denoted, respectively, by $\mathcal{A}_{\alpha B}^A$ and $\mathcal{A}_{\alpha b}^a$, that in our conventions are anti-hermitian matrices. The index $\alpha = 1, \dots, 4n_H$ is a curved index on the quaternionic manifold. The field-strengths of the connections are

$$\begin{aligned} \mathcal{F}_{\alpha\beta}^A{}_B &= \partial_\alpha \mathcal{A}_{\beta B}^A - \partial_\beta \mathcal{A}_{\alpha B}^A + [\mathcal{A}_\alpha, \mathcal{A}_\beta]^A{}_B \quad , \\ \mathcal{F}_{\alpha\beta}^a{}_b &= \partial_\alpha \mathcal{A}_{\beta b}^a - \partial_\beta \mathcal{A}_{\alpha b}^a + [\mathcal{A}_\alpha, \mathcal{A}_\beta]^a{}_b \quad , \end{aligned} \quad (2.6)$$

where $\partial_\alpha = \partial/\partial\phi^\alpha$. The request that the vielbein $V_\alpha^{aA}(\phi)$ be covariantly constant gives the following relations [16]:

$$\begin{aligned} V_{aA}^\alpha V_{bB}^\beta g_{\alpha\beta} &= \Omega_{ab} \epsilon_{AB} \quad , \\ V_{aA}^\alpha V^{\beta bA} + V_{aA}^\beta V^{\alpha bA} &= \frac{1}{n_H} g^{\alpha\beta} \delta_b^a \quad , \\ V_{aA}^\alpha V^{\beta aB} + V_{aA}^\beta V^{\alpha aB} &= g^{\alpha\beta} \delta_B^A \quad , \end{aligned} \quad (2.7)$$

where Ω_{ab} is the antisymmetric invariant tensor of $USp(2n_H)$. The raising and lowering conventions are collected in the appendix. The field-strength of the $USp(2)$ connection

$\mathcal{A}_{\alpha B}^A$ is naturally constructed in terms of V_{α}^{aA} by the relation:

$$\mathcal{F}_{\alpha\beta AB} = V_{\alpha a A} V_{\beta b}^a + V_{\alpha a B} V_{\beta}^a{}_A \quad , \quad (2.8)$$

and then the cyclic identity for the internal curvature tensor implies that the field-strength of the $USp(2n_H)$ connection $\mathcal{A}_{\alpha b}^a$ has the form

$$\mathcal{F}_{\alpha\beta ab} = V_{\alpha a A} V_{\beta b}^A + V_{\alpha b A} V_{\beta a}^A + \Omega_{abcd} V_{\alpha}^{dA} V_{\beta A}^c \quad , \quad (2.9)$$

where Ω_{abcd} is totally symmetric in its indices [16].

Now, assuming that the scalars parametrize the coset manifold $USp(2, 2n_H)/USp(2) \times USp(2n_H)$, we describe the gauging of the hypermultiplets under the group $USp(2) \times USp(2n_H)$ [12], that is the maximal compact subgroup of the isometry group. We denote the gauge fields of this group by A_{μ}^i and A_{μ}^I , where i and I take values in the adjoint representation of $USp(2)$ and $USp(2n_H)$, and the corresponding field-strengths are

$$\begin{aligned} F_{\mu\nu}^i &= \partial_{\mu} A_{\nu}^i - \partial_{\nu} A_{\mu}^i + \epsilon^{ijk} A_{\mu}^j A_{\nu}^k \quad , \\ F_{\mu\nu}^I &= \partial_{\mu} A_{\nu}^I - \partial_{\nu} A_{\mu}^I + f^{IJK} A_{\mu}^J A_{\nu}^K \quad , \end{aligned} \quad (2.10)$$

where ϵ^{ijk} and f^{IJK} are the structure constants of $USp(2)$ and $USp(2n_H)$. Under the gauge transformations

$$\delta A_{\mu}^i = D_{\mu} \Lambda^i \quad , \quad \delta A_{\mu}^I = D_{\mu} \Lambda^I \quad (2.11)$$

the scalars transform as

$$\delta \phi^{\alpha} = \Lambda^i \xi^{\alpha i} + \Lambda^I \xi^{\alpha I} \quad , \quad (2.12)$$

where $\xi^{\alpha i}$ and $\xi^{\alpha I}$ are the Killing vectors corresponding to the $USp(2)$ and $USp(2n_H)$ isometries. The covariant derivative for the scalars is then

$$D_{\mu} \phi^{\alpha} = \partial_{\mu} \phi^{\alpha} - A_{\mu}^i \xi^{\alpha i} - A_{\mu}^I \xi^{\alpha I} \quad . \quad (2.13)$$

One can correspondingly define the covariant derivatives for the spinors in a natural way, adding the composite connections $D_{\mu} \phi^{\alpha} \mathcal{A}_{\alpha}$. For instance, the covariant derivative for the hyperinos Ψ^a will contain the connections $D_{\mu} \phi^{\alpha} \mathcal{A}_{\alpha b}^a$, while the covariant derivative

for the gravitino and the tensorinos will contain the connections $D_\mu \phi^\alpha \mathcal{A}_\alpha^A$. The covariant derivatives for the gauginos $\lambda^{iA}, \lambda^{IA}$ are

$$\begin{aligned} D_\mu \lambda^{iA} &= \partial_\mu \lambda^{iA} + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \lambda^{iA} + D_\mu \phi^\alpha \mathcal{A}_{\alpha B}^A \lambda^{iB} + \epsilon^{ijk} A_\mu^j \lambda^{kA} \quad , \\ D_\mu \lambda^{IA} &= \partial_\mu \lambda^{IA} + \frac{1}{4} \omega_{\mu mn} \gamma^{mn} \lambda^{IA} + D_\mu \phi^\alpha \mathcal{A}_{\alpha B}^A \lambda^{IB} + f^{IJK} A_\mu^J \lambda^{KA} \quad . \end{aligned} \quad (2.14)$$

Notice that the gravitino, the tensorinos and the hyperinos are not coupled to the gauge vectors through terms that do not contain the hyper-scalars.

We now proceed to the construction of the model. We assume that the gauge group has the form $G = \prod_z G_z$, with G_z semi-simple. The scalars in the hypermultiplets are charged with respect to $G_1 = USp(2)$ and $G_2 = USp(2n_H)$. The field-strengths of the 2-forms $B_{\mu\nu}^r$ are

$$H_{\mu\nu\rho}^r = 3\partial_{[\mu} B_{\nu\rho]}^r + c^{rz} \omega_{\mu\nu\rho}^z \quad , \quad (2.15)$$

where c^{rz} are constants and ω^z are the Chern-Simons 3-forms:

$$\omega^z = \text{tr}_z(AdA + \frac{2}{3}A^3) \quad . \quad (2.16)$$

These 3-form field-strengths satisfy (anti)self-duality conditions, that to lowest order in the fermi fields are

$$G_{rs} H^{s\mu\nu\rho} = \frac{1}{6e} \epsilon^{\mu\nu\rho\sigma\delta\tau} H_{r\sigma\delta\tau} \quad , \quad (2.17)$$

where $G_{rs} = v_r v_s + x_r^M x_s^M$. Gauge invariance of H^r requires that B^r transform under vector gauge transformations according to

$$\delta B^r = -c^{rz} \text{tr}_z(\Lambda dA) \quad . \quad (2.18)$$

To lowest order in the fermi fields, we reproduce the construction of [6], adding the hypermultiplet couplings. The equations for all fields, with the exception of the 2-forms, can be obtained from the lagrangian

$$\begin{aligned} e^{-1} \mathcal{L} &= -\frac{1}{4} R + \frac{1}{12} G_{rs} H^{r\mu\nu\rho} H_{\mu\nu\rho}^s - \frac{1}{4} \partial_\mu v^r \partial^\mu v_r + \frac{1}{2} v_r c^{rz} \text{tr}_z(F_{\mu\nu} F^{\mu\nu}) \\ &+ \frac{1}{8e} \epsilon^{\mu\nu\rho\sigma\delta\tau} B_{\mu\nu}^r c_r^z \text{tr}_z(F_{\rho\sigma} F_{\delta\tau}) + \frac{1}{2} g_{\alpha\beta}(\phi) D_\mu \phi^\alpha D^\mu \phi^\beta \\ &+ \frac{1}{4v_r c^{r1}} \mathcal{A}_{\alpha B}^A \mathcal{A}_{\beta A}^B \xi^{\alpha i} \xi^{\beta i} + \frac{1}{4v_r c^{r2}} \mathcal{A}_{\alpha B}^A \mathcal{A}_{\beta A}^B \xi^{\alpha I} \xi^{\beta I} \end{aligned}$$

$$\begin{aligned}
& -\frac{i}{2}(\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho) - \frac{i}{2}v_r H^{r\mu\nu\rho}(\bar{\psi}_\mu \gamma_\nu \psi_\rho) + \frac{i}{2}(\bar{\chi}^M \gamma^\mu D_\mu \chi^M) \\
& -\frac{i}{24}v_r H_{\mu\nu\rho}^r(\bar{\chi}^M \gamma^{\mu\nu\rho} \chi^M) + \frac{1}{2}x_r^M \partial_\nu v^r(\bar{\psi}_\mu \gamma^\nu \gamma^\mu \chi^M) - \frac{1}{2}x_r^M H^{r\mu\nu\rho}(\bar{\psi}_\mu \gamma_{\nu\rho} \chi^M) \\
& +\frac{i}{2}(\bar{\Psi}_a \gamma^\mu D_\mu \Psi^a) + \frac{i}{24}v_r H_{\mu\nu\rho}^r(\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) - V_\alpha^{aA} D_\nu \phi^\alpha(\bar{\psi}_{\mu A} \gamma^\nu \gamma^\mu \Psi_a) \\
& -iv_r c^{rz} tr_z(\bar{\lambda} \gamma^\mu D_\mu \lambda) - \frac{i}{\sqrt{2}}v_r c^{rz} tr_z[F_{\nu\rho}(\bar{\psi}_\mu \gamma^{\nu\rho} \gamma^\mu \lambda)] \\
& -\frac{1}{\sqrt{2}}x_r^M c^{rz} tr_z[F_{\mu\nu}(\bar{\chi}^M \gamma^{\mu\nu} \lambda)] + \frac{i}{12}c_r^z H_{\mu\nu\rho}^r tr_z(\bar{\lambda} \gamma^{\mu\nu\rho} \lambda) \\
& -\sqrt{2}V_\alpha^{aA}[\xi^{\alpha i}(\bar{\lambda}_A^i \Psi_a) + \xi^{\alpha I}(\bar{\lambda}_A^I \Psi_a)] + \frac{i}{\sqrt{2}}\mathcal{A}_{\alpha B}^A[\xi^{\alpha i}(\bar{\lambda}_A^i \gamma^\mu \psi_\mu^B) + \xi^{\alpha I}(\bar{\lambda}_A^I \gamma^\mu \psi_\mu^B)] \\
& +\frac{1}{\sqrt{2}}\mathcal{A}_{\alpha B}^A[\frac{x_r^M c^{r1}}{v_s c^{s1}}\xi^{\alpha i}(\bar{\lambda}_A^i \chi^{MB}) + \frac{x_r^M c^{r2}}{v_s c^{s2}}\xi^{\alpha I}(\bar{\lambda}_A^I \chi^{MB})] \quad , \tag{2.19}
\end{aligned}$$

after imposing the (anti)self-duality conditions. With this prescription, its variation under the supersymmetry transformations

$$\begin{aligned}
\delta e_\mu^m &= -i(\bar{\epsilon} \gamma^m \psi_\mu) \quad , \\
\delta B_{\mu\nu}^r &= iv^r(\bar{\psi}_{[\mu} \gamma_{\nu]} \epsilon) + \frac{1}{2}x_r^M(\bar{\chi}^M \gamma_{\mu\nu} \epsilon) + 2c^{rz} tr_z(A_{[\mu} \delta A_{\nu]}) \quad , \\
\delta v_r &= x_r^M(\bar{\epsilon} \chi^M) \quad , \quad \delta x_r^M = v_r(\bar{\epsilon} \chi^M) \quad , \\
\delta \phi^\alpha &= V_{aA}^\alpha(\bar{\epsilon}^A \Psi^a) \quad , \\
\delta A_\mu &= -\frac{i}{\sqrt{2}}(\bar{\epsilon} \gamma_\mu \lambda) \quad , \\
\delta \psi_\mu^A &= D_\mu \epsilon^A + \frac{1}{4}v_r H_{\mu\nu\rho}^r \gamma^{\nu\rho} \epsilon^A \quad , \\
\delta \chi^{MA} &= \frac{i}{2}x_r^M \partial_\mu v^r \gamma^\mu \epsilon^A + \frac{i}{12}x_r^M H_{\mu\nu\rho}^r \gamma^{\mu\nu\rho} \epsilon^A \quad , \\
\delta \Psi^a &= i\gamma^\mu \epsilon_A V_\alpha^{aA} D_\mu \phi^\alpha \quad , \\
\delta \lambda^A &= -\frac{1}{2\sqrt{2}}F_{\mu\nu} \gamma^{\mu\nu} \epsilon^A \quad (z \neq 1, 2) \quad , \\
\delta \lambda^{iA} &= -\frac{1}{2\sqrt{2}}F_{\mu\nu}^i \gamma^{\mu\nu} \epsilon^A - \frac{1}{\sqrt{2}v_r c^{r1}}\mathcal{A}_{\alpha B}^A \xi^{\alpha i} \epsilon^B \\
\delta \lambda^{IA} &= -\frac{1}{2\sqrt{2}}F_{\mu\nu}^I \gamma^{\mu\nu} \epsilon^A - \frac{1}{\sqrt{2}v_r c^{r2}}\mathcal{A}_{\alpha B}^A \xi^{\alpha I} \epsilon^B \tag{2.20}
\end{aligned}$$

gives the supersymmetry anomaly

$$\mathcal{A}_\epsilon = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau} c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu A_\nu) tr_{z'}(F_{\rho\sigma} F_{\delta\tau}) - \frac{1}{6}\epsilon^{\mu\nu\rho\sigma\delta\tau} c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu F_{\nu\rho}) \omega_{\sigma\delta\tau}^{z'} \quad , \tag{2.21}$$

related by the Wess-Zumino conditions to the consistent gauge anomaly

$$\mathcal{A}_\Lambda = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau} c_r^z c^{rz'} tr_z(\Lambda \partial_\mu A_\nu) tr_{z'}(F_{\rho\sigma} F_{\delta\tau}) \quad . \tag{2.22}$$

Notice the presence in the lagrangian of the scalar potential

$$V(\phi) = -\frac{1}{4v_r c^{r1}} \mathcal{A}_{\alpha B}^A \mathcal{A}_{\beta A}^B \xi^{\alpha i} \xi^{\beta i} - \frac{1}{4v_r c^{r2}} \mathcal{A}_{\alpha B}^A \mathcal{A}_{\beta A}^B \xi^{\alpha I} \xi^{\beta I} \quad . \quad (2.23)$$

As in rather more conventional gauged models, the potential contains interesting informations, and it may be very instructive to study its extrema in special cases.

We now want to extend the results to all orders in the fermi fields. First of all, we define the supercovariant quantities

$$\begin{aligned} \hat{\omega}_{\mu\nu\rho} &= \omega_{\mu\nu\rho}^0 - \frac{i}{2}(\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\rho \psi_\mu + \bar{\psi}_\rho \gamma_\mu \psi_\nu) \quad , \\ \hat{H}_{\mu\nu\rho}^r &= H_{\mu\nu\rho}^r - \frac{1}{2}x^{Mr}(\bar{\chi}^M \gamma_{\mu\nu} \psi_\rho + \bar{\chi}^M \gamma_{\nu\rho} \psi_\mu + \bar{\chi}^M \gamma_{\rho\mu} \psi_\nu) \\ &\quad - \frac{i}{2}v^r(\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\rho \psi_\mu + \bar{\psi}_\rho \gamma_\mu \psi_\nu) \quad , \\ \partial_\mu \hat{v}^r &= \partial_\mu v^r - x^{Mr}(\bar{\chi}^M \psi_\mu) \quad , \\ D_\mu \hat{\phi}^\alpha &= D_\mu \phi^\alpha - V_{aA}^\alpha(\bar{\psi}_\mu^A \Psi^a) \quad , \\ \hat{F}_{\mu\nu} &= F_{\mu\nu} + \frac{i}{\sqrt{2}}(\bar{\lambda} \gamma_\mu \psi_\nu) - \frac{i}{\sqrt{2}}(\bar{\lambda} \gamma_\nu \psi_\mu) \quad , \end{aligned} \quad (2.24)$$

and require that the transformation rules for the fermi fields be supercovariant. All fermionic terms in the supersymmetry transformations of the fermi fields that are not determined by supercovariance are then obtained requiring the closure of the supersymmetry algebra on bose and fermi fields. Moreover, since the supersymmetry algebra on the fermi fields closes only on-shell, in this way one can determine the complete fermionic field equations, and from these the complete lagrangian, up to some subtleties related to the (anti)self-dual forms, that will be described in section 4.

The complete supersymmetry transformations of the fermi fields are

$$\begin{aligned} \delta\psi_\mu^A &= D_\mu(\hat{\omega})\epsilon^A + \frac{1}{4}v_r \hat{H}_{\mu\nu\rho}^r \gamma^{\nu\rho} \epsilon^A - \frac{3i}{8}\gamma_\mu \chi^{MA}(\bar{\epsilon}\chi^M) - \frac{i}{8}\gamma^\nu \chi^{MA}(\bar{\epsilon}\gamma_{\mu\nu}\chi^M) \\ &\quad + \frac{i}{16}\gamma_{\mu\nu\rho} \chi^{MA}(\bar{\epsilon}\gamma^{\nu\rho}\chi^M) + \frac{9i}{8}v_r c^{rz} tr_z[\lambda^A(\bar{\epsilon}\gamma_\mu\lambda)] - \frac{i}{8}v_r c^{rz} tr_z[\gamma_{\mu\nu}\lambda^A(\bar{\epsilon}\gamma^\nu\lambda)] \\ &\quad + \frac{i}{16}v_r c^{rz} tr_z[\gamma^{\nu\rho}\lambda^A(\bar{\epsilon}\gamma_{\mu\nu\rho}\lambda)] - \delta\phi^\alpha \mathcal{A}_{\alpha B}^A \psi_\mu^B \quad , \\ \delta\chi^{MA} &= \frac{i}{2}x_r^M(\partial_\mu \hat{v}^r)\gamma^\mu \epsilon^A + \frac{i}{12}x_r^M \hat{H}_{\mu\nu\rho}^r \gamma^{\mu\nu\rho} \epsilon^A \\ &\quad - \frac{1}{2}x_r^M c^{rz} tr_z[\gamma_\mu \lambda^A(\bar{\epsilon}\gamma^\mu\lambda)] - \delta\phi^\alpha \mathcal{A}_{\alpha B}^A \chi^{MB} \quad , \end{aligned}$$

$$\begin{aligned}
\delta\Psi^a &= i\gamma^\mu\epsilon_A V_\alpha^{aA} D_\mu \hat{\phi}^\alpha - \delta\phi^\alpha \mathcal{A}_{\alpha b}^a \Psi^b \quad , \\
\delta\lambda^A &= -\frac{1}{2\sqrt{2}} \hat{F}_{\mu\nu} \gamma^{\mu\nu} \epsilon^A - \frac{x_r^M c^{rz}}{2v_s c^{sz}} (\bar{\chi}^M \lambda) \epsilon^A - \frac{x_r^M c^{rz}}{4v_s c^{sz}} (\bar{\chi}^M \epsilon) \lambda^A \\
&\quad + \frac{x_r^M c^{rz}}{8v_s c^{sz}} (\bar{\chi}^M \gamma_{\mu\nu} \epsilon) \gamma^{\mu\nu} \lambda^A - \delta\phi^\alpha \mathcal{A}_{\alpha B}^A \lambda^B \quad (z \neq 1, 2) \quad , \\
\delta\lambda^{iA} &= -\frac{1}{2\sqrt{2}} \hat{F}_{\mu\nu}^i \gamma^{\mu\nu} \epsilon^A - \frac{x_r^M c^{r1}}{2v_s c^{s1}} (\bar{\chi}^M \lambda^i) \epsilon^A - \frac{x_r^M c^{r1}}{4v_s c^{s1}} (\bar{\chi}^M \epsilon) \lambda^{iA} \\
&\quad + \frac{x_r^M c^{r1}}{8v_s c^{s1}} (\bar{\chi}^M \gamma_{\mu\nu} \epsilon) \gamma^{\mu\nu} \lambda^{iA} - \delta\phi^\alpha \mathcal{A}_{\alpha B}^A \lambda^{iB} - \frac{1}{\sqrt{2}v_r c^{r1}} \mathcal{A}_{\alpha B}^A \xi^{\alpha i} \epsilon^B \quad , \\
\delta\lambda^{IA} &= -\frac{1}{2\sqrt{2}} \hat{F}_{\mu\nu}^I \gamma^{\mu\nu} \epsilon^A - \frac{x_r^M c^{r2}}{2v_s c^{s2}} (\bar{\chi}^M \lambda^I) \epsilon^A - \frac{x_r^M c^{r2}}{4v_s c^{s2}} (\bar{\chi}^M \epsilon) \lambda^{IA} \\
&\quad + \frac{x_r^M c^{r2}}{8v_s c^{s2}} (\bar{\chi}^M \gamma_{\mu\nu} \epsilon) \gamma^{\mu\nu} \lambda^{IA} - \delta\phi^\alpha \mathcal{A}_{\alpha B}^A \lambda^{IB} - \frac{1}{\sqrt{2}v_r c^{r2}} \mathcal{A}_{\alpha B}^A \xi^{\alpha I} \epsilon^B \quad . \quad (2.25)
\end{aligned}$$

One can compute the commutators of two supersymmetry transformations on the bose fields using these relations, and show that they generate the local symmetries:

$$[\delta_1, \delta_2] = \delta_{gct} + \delta_{Lorentz} + \delta_{susy} + \delta_{tens} + \delta_{gauge} + \delta_{SO(n)} \quad , \quad (2.26)$$

where the parameters of generic coordinate, local Lorentz, supersymmetry, tensor gauge, vector gauge and composite $SO(n)$ transformations are respectively

$$\begin{aligned}
\xi_\mu &= -i(\bar{\epsilon}_1 \gamma_\mu \epsilon_2) \quad , \\
\Omega^{mn} &= -i\xi^\mu (\hat{\omega}_\mu^{mn} - v_r \hat{H}_\mu^{rmn}) - \frac{1}{2} [(\bar{\chi}^M \epsilon_1)(\bar{\epsilon}_2 \gamma^{mn} \chi^M) - (\bar{\chi}^M \epsilon_2)(\bar{\epsilon}_1 \gamma^{mn} \chi^M)] \\
&\quad - v_r c^{rz} tr_z [(\bar{\epsilon}_1 \gamma^m \lambda)(\bar{\epsilon}_2 \gamma^n \lambda) - (\bar{\epsilon}_2 \gamma^m \lambda)(\bar{\epsilon}_1 \gamma^n \lambda)] \quad , \\
\zeta^A &= \xi^\mu \psi_\mu^A + V_{aC}^\alpha \mathcal{A}_{\alpha B}^A \epsilon_2^B (\bar{\epsilon}_1^C \Psi^a) - V_{aC}^\alpha \mathcal{A}_{\alpha B}^A \epsilon_1^B (\bar{\epsilon}_2^C \Psi^a) \quad , \\
\Lambda_\mu^r &= -\frac{1}{2} v^r \xi_\mu - \xi^\nu B_{\mu\nu}^r \quad , \\
\Lambda &= \xi^\mu A_\mu \quad , \\
A^{MN} &= \xi^\mu x^{Mr} (\partial_\mu x_r^N) + (\bar{\chi}^M \epsilon_2)(\bar{\chi}^N \epsilon_1) - (\bar{\chi}^M \epsilon_1)(\bar{\chi}^N \epsilon_2) \quad . \quad (2.27)
\end{aligned}$$

In order to prove this result, one has to use the (anti)self-duality condition for the tensor fields, that to all orders in the fermi fields is

$$G_{rs} \hat{\mathcal{H}}_{\mu\nu\rho}^s = \frac{1}{6e} \epsilon_{\mu\nu\rho\sigma\delta\tau} \hat{\mathcal{H}}_r^{\sigma\delta\tau} \quad (2.28)$$

in terms of the 3-forms [13]

$$\hat{\mathcal{H}}_{\mu\nu\rho}^r = \hat{H}_{\mu\nu\rho}^r - \frac{i}{8} v^r (\bar{\chi}^M \gamma_{\mu\nu\rho} \chi^M) + \frac{i}{8} v^r (\bar{\Psi}_a \gamma_{\mu\nu\rho} \Psi^a) - \frac{i}{4} c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) \quad . \quad (2.29)$$

Requiring that the commutator of two supersymmetry transformations on the fermi fields close on-shell then determines the complete fermi field equations. The equations obtained in this way are

$$\begin{aligned}
& -i\gamma^{\mu\nu\rho}D_\nu(\hat{\omega})\psi_\rho^A - \frac{i}{4}v_r\hat{H}_{\nu\sigma\delta}^r\gamma^{\mu\nu\rho}\gamma^{\sigma\delta}\psi_\rho^A - \frac{1}{12}x_r^M\hat{H}^{r\nu\rho\sigma}\gamma_{\nu\rho\sigma}\gamma^\mu\chi^{MA} \\
& + \frac{1}{2}x_r^M(\partial_\nu\hat{v}^r)\gamma^\nu\gamma^\mu\chi^{MA} + \frac{3}{2}\gamma^{\mu\nu}\chi^{MA}(\bar{\chi}^M\psi_\nu) - \frac{1}{4}\gamma^{\mu\nu}\chi^{MA}(\bar{\chi}^M\gamma_{\nu\rho}\psi^\rho) \\
& + \frac{1}{4}\gamma_{\nu\rho}\chi^{MA}(\bar{\chi}^M\gamma^{\mu\nu}\psi^\rho) - \frac{1}{2}\chi^{MA}(\bar{\chi}^M\gamma^{\mu\nu}\psi_\nu) + iv_rc^{rz}tr_z[-\frac{1}{\sqrt{2}}\gamma^{\nu\rho}\gamma^\mu\lambda^A\hat{F}_{\nu\rho} \\
& + \frac{3i}{4}\gamma^{\mu\nu\rho}\lambda^A(\bar{\psi}_\nu\gamma_\rho\lambda) - \frac{i}{2}\gamma^\mu\lambda^A(\bar{\psi}_\nu\gamma^\nu\lambda) + \frac{i}{2}\gamma^\nu\lambda^A(\bar{\psi}_\nu\gamma^\mu\lambda) + \frac{i}{4}\gamma_\rho\lambda^A(\bar{\psi}_\nu\gamma^{\mu\nu\rho}\lambda)] \\
& + \frac{i}{2}x_r^Mc^{rz}tr_z[\gamma_\nu\lambda^A(\bar{\chi}^M\gamma^\nu\gamma^\mu\lambda)] - V_\alpha^{aA}D_\nu\hat{\phi}^\alpha\gamma^\nu\gamma^\mu\Psi_a \\
& + \frac{i}{\sqrt{2}}[\mathcal{A}_{\alpha B}^a\xi^{\alpha i}\gamma^\mu\lambda^{iB} + \mathcal{A}_{\alpha B}^a\xi^{\alpha I}\gamma^\mu\lambda^{IB}] = 0
\end{aligned} \tag{2.30}$$

for the gravitino,

$$\begin{aligned}
& i\gamma^\mu D_\mu(\hat{\omega})\chi^{MA} - \frac{i}{12}v_r\hat{H}_{\mu\nu\rho}^r\gamma^{\mu\nu\rho}\chi^{MA} + \frac{1}{12}x_r^M\hat{H}_{\mu\nu\rho}^r\gamma^\sigma\gamma^{\mu\nu\rho}\psi_\sigma^A + \frac{1}{2}x_r^M(\partial_\nu\hat{v}^r)\gamma^\mu\gamma^\nu\psi_\mu^A \\
& - \frac{1}{\sqrt{2}}x_r^Mc^{rz}tr_z[\hat{F}_{\mu\nu}\gamma^{\mu\nu}\lambda^A] + \frac{i}{2}x_r^Mc^{rz}tr_z[\gamma^\mu\gamma^\nu\lambda^A(\bar{\psi}_\mu\gamma_\nu\lambda)] + \frac{1}{2}\gamma^\mu\chi^{NA}(\bar{\chi}^N\gamma_\mu\chi^M) \\
& + \frac{3}{8}v_rc^{rz}tr_z[(\bar{\chi}^M\gamma_{\mu\nu}\lambda)\gamma^{\mu\nu}\lambda^A] + \frac{1}{4}v_rc^{rz}tr_z[(\bar{\chi}^M\lambda)\lambda^A] \\
& + \frac{3}{2}\frac{x_r^Mc^{rz}x_s^Nc^{sz}}{v_tc^{tz}}tr_z[(\bar{\chi}^N\lambda)\lambda^A] - \frac{1}{4}\frac{x_r^Mc^{rz}x_s^Nc^{sz}}{v_tc^{tz}}tr_z[(\bar{\chi}^N\gamma_{\mu\nu}\lambda)\gamma^{\mu\nu}\lambda^A] \\
& - \frac{x_r^Mc^{r1}}{\sqrt{2}v_sc^{s1}}\mathcal{A}_{\alpha B}^A\xi^{\alpha i}\lambda^{iB} - \frac{x_r^Mc^{r2}}{\sqrt{2}v_sc^{s2}}\mathcal{A}_{\alpha B}^A\xi^{\alpha I}\lambda^{IB} = 0
\end{aligned} \tag{2.31}$$

for the tensorinos,

$$\begin{aligned}
& i\gamma^\mu D_\mu(\hat{\omega})\Psi^a + \frac{i}{12}v_r\hat{H}_{\mu\nu\rho}^r\gamma^{\mu\nu\rho}\Psi^a + \gamma^\mu\gamma^\nu\psi_{\mu A}V_\alpha^{aA}D_\nu\hat{\phi}^\alpha + \frac{1}{48}v_rc^{rz}tr_z(\bar{\lambda}\gamma_{\mu\nu\rho}\lambda)\gamma^{\mu\nu\rho}\Psi^a \\
& + \frac{1}{12}\Omega^{abcd}\gamma^\mu\Psi_b(\bar{\Psi}_c\gamma_\mu\Psi_d) + \sqrt{2}V_\alpha^{aA}[\xi^{\alpha i}\lambda_A^i + \xi^{\alpha I}\lambda_A^I] = 0
\end{aligned} \tag{2.32}$$

for the hyperinos. More care is needed in order to derive the equations for the gauginos, since the $c_r^zc^{rz'}$ terms in the commutator of two supersymmetry transformations are

$$\begin{aligned}
& \frac{c_r^zc^{rz'}}{v_sc^{sz}}tr_{z'}[\frac{1}{4}(\bar{\epsilon}_1\gamma_\mu\lambda')(\bar{\epsilon}_2\gamma_\nu\lambda')\gamma^{\mu\nu}\lambda^A - \frac{1}{4}(\bar{\lambda}\gamma_\mu\lambda')(\bar{\epsilon}_1\gamma^\mu\lambda')\epsilon_2^A - (1 \leftrightarrow 2) \\
& - \frac{1}{16}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)(\bar{\lambda}'\gamma_{\mu\nu\rho}\lambda')\gamma^{\nu\rho}\lambda^A] \quad .
\end{aligned} \tag{2.33}$$

If one allows for the term

$$i\alpha c_r^zc^{rz'}tr_{z'}[(\bar{\lambda}\gamma_\mu\lambda')\gamma^\mu\lambda'^A] \tag{2.34}$$

in the gaugino field equation, then what remains of eq. (2.33) is

$$\begin{aligned}
\delta_{extra(\alpha)}\lambda^A &= \frac{c_r^z c^{rz'}}{v_s c^{sz}} tr_{z'} \left[\frac{1}{4} (\bar{\epsilon}_1 \gamma_\mu \lambda') (\bar{\epsilon}_2 \gamma_\nu \lambda') \gamma^{\mu\nu} \lambda^A \right. \\
&\quad + \frac{\alpha}{2} (\bar{\lambda} \gamma_\mu \lambda') (\bar{\epsilon}_1 \gamma_\nu \lambda') \gamma^{\mu\nu} \epsilon_2^A - \frac{\alpha}{16} (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda') (\bar{\epsilon}_1 \gamma^\rho \lambda') \gamma^{\mu\nu} \epsilon_2^A \\
&\quad - \frac{\alpha}{16} (\bar{\lambda} \gamma_\rho \lambda') (\bar{\epsilon}_1 \gamma^{\mu\nu\rho} \lambda') \gamma_{\mu\nu} \epsilon_2^A - \frac{1-\alpha}{4} (\bar{\lambda} \gamma_\mu \lambda') (\bar{\epsilon}_1 \gamma^\mu \lambda') \epsilon_2^A - (1 \leftrightarrow 2) \\
&\quad \left. - \frac{1-\alpha}{16} (\bar{\epsilon}_1 \gamma^\rho \epsilon_2) (\bar{\lambda}' \gamma_{\mu\nu\rho} \lambda') \gamma^{\mu\nu} \lambda^A \right] \quad . \tag{2.35}
\end{aligned}$$

As explained in [7], no choice of α can eliminate all these terms, that play the role of a central charge felt only by the gauginos. This is the “classical” realization of a general feature: anomalies in current conservations are accompanied by related anomalies in current commutators [17]. When this is properly taken into account, the field equations for the gauginos are

$$\begin{aligned}
& i v_r c^{rz} \gamma^\mu D_\mu(\hat{\omega}) \lambda^A + \frac{i}{2} (\partial_\mu \hat{v}_r) c^{rz} \gamma^\mu \lambda^A + \frac{i}{2\sqrt{2}} v_r c^{rz} \hat{F}_{\nu\rho} \gamma^\mu \gamma^{\nu\rho} \psi_\mu^A - \frac{1}{2\sqrt{2}} x_r^M c^{rz} \hat{F}_{\mu\nu} \gamma^{\mu\nu} \chi^{MA} \\
& + \frac{i}{12} x_r^M c^{rz} x_s^M \hat{H}_{\mu\nu\rho}^s \gamma^{\mu\nu\rho} \lambda^A + \frac{i}{2} x_r^M c^{rz} (\bar{\chi}^M \lambda) \gamma^\mu \psi_\mu^A + \frac{i}{4} x_r^M c^{rz} (\bar{\chi}^M \psi_\mu) \gamma^\mu \lambda^A \\
& - \frac{i}{8} x_r^M c^{rz} (\bar{\chi}^M \gamma_{\nu\rho} \psi_\mu) \gamma^{\mu\nu\rho} \lambda^A - \frac{i}{4} x_r^M c^{rz} (\bar{\chi}^M \gamma_{\mu\nu} \psi^\mu) \gamma^\nu \lambda^A - \frac{1}{8} v_r c^{rz} (\bar{\lambda} \chi^M) \chi^{MA} \\
& - \frac{3}{16} v_r c^{rz} (\bar{\lambda} \gamma_{\mu\nu} \chi^M) \gamma^{\mu\nu} \chi^{MA} - \frac{3}{4} \frac{x_r^M c^{rz} x_s^N c^{sz}}{v_t c^{tz}} (\bar{\lambda} \chi^M) \chi^{NA} \\
& + \frac{1}{8} \frac{x_r^M c^{rz} x_s^N c^{sz}}{v_t c^{tz}} (\bar{\lambda} \gamma_{\mu\nu} \chi^M) \gamma^{\mu\nu} \chi^{NA} - \frac{1}{96} (\bar{\Psi}_a \gamma_{\mu\nu\rho} \Psi^a) \gamma^{\mu\nu\rho} \lambda^A \\
& + v_r v_s c^{rz} c^{sz'} tr_{z'} [(\bar{\lambda} \gamma_\mu \lambda') \gamma^\mu \lambda'^A] - \alpha c_r^z c^{rz'} tr_{z'} [(\bar{\lambda} \gamma_\mu \lambda') \gamma^\mu \lambda'^A] = 0 \quad . \tag{2.36}
\end{aligned}$$

Actually to the left-hand side of this equation, valid for the case $z \neq 1, 2$, one has to add the terms

$$- \sqrt{2} V_\alpha^{aA} \xi^{\alpha i, I} \Psi_a + \frac{i}{\sqrt{2}} \mathcal{A}_{\alpha B}^A \xi^{\alpha i, I} \gamma^\mu \psi_\mu^B + \frac{x_r^M c^{r1,2}}{\sqrt{2} v_s c^{s1,2}} \mathcal{A}_{\alpha B}^A \xi^{\alpha i, I} \chi^{MB} \tag{2.37}$$

in the two remaining cases, i.e. for λ^i and λ^I .

Having obtained the complete fermionic field equations, one can add to eq. (2.19) all the terms quartic in the fermi fields, thus obtaining the complete lagrangian

$$\begin{aligned}
e^{-1} \mathcal{L} &= -\frac{1}{4} R + \frac{1}{12} G_{rs} H^{r\mu\nu\rho} H_{\mu\nu\rho}^s - \frac{1}{4} \partial_\mu v^r \partial^\mu v_r + \frac{1}{2} g_{\alpha\beta}(\phi) D_\mu \phi^\alpha D^\mu \phi^\beta \\
&\quad + \frac{1}{2} v_r c^{rz} tr_z (F_{\mu\nu} F^{\mu\nu}) + \frac{1}{8e} \epsilon^{\mu\nu\rho\sigma\delta\tau} c_r^z B_{\mu\nu}^r tr_z (F_{\rho\sigma} F_{\delta\tau})
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4v_r c^{r1}} \mathcal{A}_{\alpha B}^A \mathcal{A}_{\beta A}^B \xi^{\alpha i} \xi^{\beta i} + \frac{1}{4v_r c^{r2}} \mathcal{A}_{\alpha B}^A \mathcal{A}_{\beta A}^B \xi^{\alpha I} \xi^{\beta I} \\
& - \frac{i}{2} (\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu [\frac{1}{2}(\omega + \hat{\omega})] \psi_\rho) - \frac{i}{8} v_r [H + \hat{H}]^{r\mu\nu\rho} (\bar{\psi}_\mu \gamma_\nu \psi_\rho) \\
& + \frac{i}{48} v_r [H + \hat{H}]_{\rho\sigma\delta}^r (\bar{\psi}_\mu \gamma^{\mu\nu\rho\sigma\delta} \psi_\nu) + \frac{i}{2} (\bar{\chi}^M \gamma^\mu D_\mu (\hat{\omega}) \chi^M) \\
& - \frac{i}{24} v_r \hat{H}_{\mu\nu\rho}^r (\bar{\chi}^M \gamma^{\mu\nu\rho} \chi^M) + \frac{1}{4} x_r^M [\partial_\nu v^r + \partial_\nu \hat{v}^r] (\bar{\psi}_\mu \gamma^\nu \gamma^\mu \chi^M) \\
& - \frac{1}{8} x_r^M [H + \hat{H}]^{r\mu\nu\rho} (\bar{\psi}_\mu \gamma_{\nu\rho} \chi^M) + \frac{1}{24} x_r^M [H + \hat{H}]^{r\mu\nu\rho} (\bar{\psi}^\sigma \gamma_{\sigma\mu\nu\rho} \chi^M) \\
& + \frac{i}{2} (\bar{\Psi}_a \gamma^\mu D_\mu (\hat{\omega}) \Psi^a) + \frac{i}{24} v_r \hat{H}_{\mu\nu\rho}^r (\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) \\
& - \frac{1}{2} V_\alpha^{aA} [D_\nu \phi^\alpha + D_\nu \hat{\phi}^\alpha] (\bar{\psi}_{\mu A} \gamma^\nu \gamma^\mu \Psi_a) \\
& - i v_r c^{rz} tr_z (\bar{\lambda} \gamma^\mu D_\mu (\hat{\omega}) \lambda) - \frac{i}{12} x_r^M x_s^M \hat{H}_{\mu\nu\rho}^r c^{sz} tr_z (\bar{\lambda} \gamma^{\mu\nu\rho} \lambda) \\
& - \frac{i}{2\sqrt{2}} v_r c^{rz} tr_z [(F + \hat{F})_{\nu\rho} (\bar{\psi}_\mu \gamma^{\nu\rho} \gamma^\mu \lambda)] - \frac{1}{\sqrt{2}} x_r^M c^{rz} tr_z [(\bar{\chi}^M \gamma^{\mu\nu} \lambda) \hat{F}_{\mu\nu}] \\
& - \sqrt{2} V_\alpha^{aA} [\xi^{\alpha i} (\bar{\lambda}_A^i \Psi_a) + \xi^{\alpha I} (\bar{\lambda}_A^I \Psi_a)] + \frac{i}{\sqrt{2}} \mathcal{A}_{\alpha B}^A [\xi^{\alpha i} (\bar{\lambda}_A^i \gamma^\mu \psi_\mu^B) + \xi^{\alpha I} (\bar{\lambda}_A^I \gamma^\mu \psi_\mu^B)] \\
& + \frac{1}{\sqrt{2}} \mathcal{A}_{\alpha B}^A [\frac{x_r^M c^{r1}}{v_s c^{s1}} \xi^{\alpha i} (\bar{\lambda}_A^i \chi^{MB}) + \frac{x_r^M c^{r2}}{v_s c^{s2}} \xi^{\alpha I} (\bar{\lambda}_A^I \chi^{MB})] \\
& + \frac{1}{8} (\bar{\chi}^M \gamma^{\mu\nu\rho} \chi^M) (\bar{\psi}_\mu \gamma_\nu \psi_\rho) - \frac{1}{8} (\bar{\chi}^M \gamma^\mu \chi^N) (\bar{\chi}^M \gamma_\mu \chi^N) \\
& + \frac{1}{8} (\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) (\psi_\mu \gamma_\nu \psi_\rho) + \frac{1}{48} \Omega^{abcd} (\bar{\Psi}_a \gamma_\mu \Psi_b) (\bar{\Psi}_c \gamma^\mu \Psi_d) \\
& - \frac{1}{16} v_r c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) (\bar{\chi}^M \gamma^{\mu\nu\rho} \chi^M) + \frac{i}{8} (\bar{\chi}^M \gamma_{\mu\nu} \psi_\rho) x_r^M c^{rz} tr_z (\bar{\lambda} \gamma^{\mu\nu\rho} \lambda) \\
& + \frac{i}{2} x_r^M c^{rz} tr_z [(\bar{\chi}^M \gamma^\mu \gamma^\nu \lambda) (\bar{\psi}_\mu \gamma_\nu \lambda)] - \frac{1}{4} (\bar{\psi}_\mu \gamma_\nu \psi_\rho) v_r c^{rz} tr_z (\bar{\lambda} \gamma^{\mu\nu\rho} \lambda) \\
& + \frac{1}{8} v_r c^{rz} tr_z [(\bar{\chi}^M \lambda) (\bar{\chi}^M \lambda)] + \frac{3}{16} v_r c^{rz} tr_z [(\bar{\chi}^M \gamma_{\mu\nu} \lambda) (\bar{\chi}^M \gamma^{\mu\nu} \lambda)] \\
& + \frac{3x_r^M c^{rz} x_s^N c^{sz}}{4v_t c^{tz}} tr_z [(\bar{\chi}^M \lambda) (\bar{\chi}^N \lambda)] - \frac{x_r^M c^{rz} x_s^N c^{sz}}{8v_t c^{tz}} tr_z [(\bar{\chi}^M \gamma_{\mu\nu} \lambda) (\bar{\chi}^N \gamma^{\mu\nu} \lambda)] \\
& - \frac{5}{96} v_r c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) (\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) - \frac{1}{2} v_r v_s c^{rz} c^{sz'} tr_{z,z'} [(\bar{\lambda} \gamma_\mu \lambda') (\bar{\lambda} \gamma^\mu \lambda')] \\
& + \frac{\alpha}{2} c^{rz} c_r^{z'} tr_{z,z'} [(\bar{\lambda} \gamma_\mu \lambda') (\bar{\lambda} \gamma^\mu \lambda')] \quad . \tag{2.38}
\end{aligned}$$

From this lagrangian, in the 1.5 order formalism and using the (anti)self-duality conditions of eqs. (2.28) and (2.29), one can obtain the remaining complete bosonic field equations. Once more, it is important to notice that this lagrangian is neither gauge invariant nor supersymmetric: its variation under gauge transformations produces the gauge anomaly of eq. (2.22), while its variation under the complete supersymmetry transformations

produces the complete supersymmetry anomaly

$$\begin{aligned}
\mathcal{A}_\epsilon &= c_r^z c^{rz'} tr_{z,z'} \left\{ -\frac{1}{4} \epsilon^{\mu\nu\rho\sigma\delta\tau} \delta_\epsilon A_\mu A_\nu F'_{\rho\sigma} F'_{\delta\tau} - \frac{1}{6} \epsilon^{\mu\nu\rho\sigma\delta\tau} \delta_\epsilon A_\mu F_{\nu\rho} \omega'_{\sigma\delta\tau} \right. \\
&\quad + \frac{ie}{2} \delta_\epsilon A_\mu F_{\nu\rho} (\bar{\lambda}' \gamma^{\mu\nu\rho} \lambda') + \frac{ie}{2} \delta_\epsilon A_\mu (\bar{\lambda} \gamma^{\mu\nu\rho} \lambda') F'_{\nu\rho} + ie \delta_\epsilon A_\mu (\bar{\lambda} \gamma_\nu \lambda') F'^{\mu\nu} \\
&\quad + \frac{e}{32} \delta_\epsilon e_\mu{}^m (\bar{\lambda} \gamma^{\mu\nu\rho} \lambda) (\bar{\lambda}' \gamma_{m\nu\rho} \lambda') - \frac{e}{2\sqrt{2}} \delta_\epsilon A_\mu (\bar{\lambda} \gamma^\mu \gamma^\nu \gamma^\rho \lambda') (\bar{\lambda}' \gamma_\nu \psi_\rho) \\
&\quad + \frac{e x_s^M c^{sz'}}{v_t c^{tz'}} \left[-\frac{3i}{2\sqrt{2}} \delta_\epsilon A_\mu (\bar{\lambda} \gamma^\mu \lambda') (\bar{\lambda}' \chi^M) - \frac{i}{4\sqrt{2}} \delta_\epsilon A_\mu (\bar{\lambda} \gamma^{\mu\nu\rho} \lambda') (\bar{\lambda}' \gamma_{\nu\rho} \chi^M) \right. \\
&\quad \left. - \frac{i}{2\sqrt{2}} \delta_\epsilon A_\mu (\bar{\lambda} \gamma_\nu \lambda') (\bar{\lambda}' \gamma^{\mu\nu} \chi^M) \right] + \frac{\alpha}{2} \delta_\epsilon [e (\bar{\lambda} \gamma_\mu \lambda') (\bar{\lambda} \gamma^\mu \lambda')] \} \\
&\quad + \frac{ie c_r^1 c^{rz}}{2 v_s c^{s1}} \mathcal{A}_{\alpha B}^A \xi^{\alpha i} tr_z [\delta_\epsilon A_\mu (\bar{\lambda}_A^i \gamma^\mu \lambda^B)] \\
&\quad + \frac{ie c_r^2 c^{rz}}{2 v_s c^{s2}} \mathcal{A}_{\alpha B}^A \xi^{\alpha I} tr_z [\delta_\epsilon A_\mu (\bar{\lambda}_A^I \gamma^\mu \lambda^B)] \ .
\end{aligned} \tag{2.39}$$

The presence of a term proportional to the parameter α in eq. (2.38) reflects the general fact that anomalies are defined up to the variation of a local functional. Gauge and supersymmetry anomalies are in general related by the Wess-Zumino consistency conditions [5]

$$\begin{aligned}
\delta_\epsilon \mathcal{A}_\Lambda &= \delta_\Lambda \mathcal{A}_\epsilon \quad , \\
\delta_{\epsilon_1} \mathcal{A}_{\epsilon_2} - \delta_{\epsilon_2} \mathcal{A}_{\epsilon_1} &= \mathcal{A}_\Lambda + \mathcal{A}_\zeta \quad .
\end{aligned} \tag{2.40}$$

What is peculiar of these six-dimensional models is the fact that the second condition closes only on-shell, and precisely on the gaugino field equations [7]. Since the inclusion of the term proportional to α in the lagrangian modifies both these equations and the supersymmetry anomaly, there must be some extra terms that permit the Wess-Zumino conditions to close on-shell for every value of α . This is precisely the role of the terms in eq. (2.35) in the commutator of two supersymmetry transformations on the gauginos, that thus can be seen as a transformation needed in order to close the Wess-Zumino conditions precisely on the field equations determined by the algebra. Since the Wess-Zumino conditions need only the equation of the gauginos, only these fields sense the additional transformation (the whole construction is explained in more detail in [7]).

3 Inclusion of abelian vectors

Up to now, we have always considered the case in which the gauge group is non-abelian. In the abelian case, the couplings can actually have a more general form, since gauge invariance allows non-diagonal kinetic and Chern-Simons terms, in which the constants c_z^r are substituted by generic symmetric matrices c_{IJ}^r , with I, J running over the various $U(1)$ factors [18]. In [11], the complete coupling of minimal six-dimensional supergravity to tensor multiplets and abelian vector multiplets was constructed. We now want to generalize it to the case in which also charged hypermultiplets are present, and therefore we will consider the gauging with respect to abelian subgroups of $USp(2) \times USp(2n_H)$. There are no subtleties when the symmetric matrices c_{IJ}^r are diagonal (or simultaneously diagonalizable), since in this situation the results of the previous section can be straightforwardly applied. We are thus interested in the case in which the c_{IJ}^r can not be simultaneously diagonalized. To this end, we will consider a model in which only these abelian gauge groups are present. The most general situation can be obtained combining the following results with those obtained in the previous section.

We denote with A_μ^I , $I = 1, \dots, m$, the set of abelian vectors, and the gauginos are correspondingly denoted by λ^{IA} . We collect here only the final results, since the construction follows the same lines as in the non-abelian case. All the field equations may then be derived from the lagrangian

$$\begin{aligned}
e^{-1}\mathcal{L} = & -\frac{1}{4}R + \frac{1}{12}G_{rs}H^{r\mu\nu\rho}H_{\mu\nu\rho}^s - \frac{1}{4}\partial_\mu v^r\partial^\mu v_r \\
& -\frac{1}{4}v_r c^{rIJ}F_{\mu\nu}^I F^{J\mu\nu} - \frac{1}{16e}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_r^{IJ}B_{\mu\nu}^r F_{\rho\sigma}^I F_{\delta\tau}^J \\
& +\frac{1}{2}g_{\alpha\beta}(\phi)D_\mu\phi^\alpha D^\mu\phi^\beta + \frac{1}{4}[(v \cdot c)^{-1}]^{IJ}\mathcal{A}_{\alpha B}^A\mathcal{A}_{\beta A}^B\xi^{\alpha I}\xi^{\beta J} \\
& -\frac{i}{2}(\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu[\frac{1}{2}(\omega + \hat{\omega})]\psi_\rho) - \frac{i}{8}v_r[H + \hat{H}]^{r\mu\nu\rho}(\bar{\psi}_\mu\gamma_\nu\psi_\rho) \\
& +\frac{i}{48}v_r[H + \hat{H}]_{\rho\sigma\delta}^r(\bar{\psi}_\mu\gamma^{\mu\nu\rho\sigma\delta}\psi_\nu) + \frac{i}{2}(\bar{\chi}^M\gamma^\mu D_\mu(\hat{\omega})\chi^M) \\
& -\frac{i}{24}v_r\hat{H}_{\mu\nu\rho}^r(\bar{\chi}^M\gamma^{\mu\nu\rho}\chi^M) + \frac{1}{4}x_r^M[\partial_\nu v^r + \partial_\nu\hat{v}^r](\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^M) \\
& -\frac{1}{8}x_r^M[H + \hat{H}]^{r\mu\nu\rho}(\bar{\psi}_\mu\gamma_{\nu\rho}\chi^M) + \frac{1}{24}x_r^M[H + \hat{H}]^{r\mu\nu\rho}(\bar{\psi}^\sigma\gamma_{\sigma\mu\nu\rho}\chi^M)
\end{aligned}$$

$$\begin{aligned}
& + \frac{i}{2}(\bar{\Psi}_a \gamma^\mu D_\mu(\hat{\omega}) \Psi^a) + \frac{i}{24} v_r \hat{H}_{\mu\nu\rho}^r (\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) \\
& - \frac{1}{2} V_\alpha^{aA} [D_\nu \phi^\alpha + D_\nu \hat{\phi}^\alpha] (\bar{\psi}_{\mu A} \gamma^\nu \gamma^\mu \Psi_a) \\
& + \frac{i}{2} v_r c^{rIJ} (\bar{\lambda}^I \gamma^\mu D_\mu(\hat{\omega}) \lambda^J) + \frac{i}{24} x_r^M x_s^M \hat{H}_{\mu\nu\rho}^r c^{sIJ} (\bar{\lambda}^I \gamma^{\mu\nu\rho} \lambda^J) \\
& + \frac{i}{4\sqrt{2}} v_r c^{rIJ} (F + \hat{F})_{\nu\rho}^I (\bar{\psi}_\mu \gamma^{\nu\rho} \gamma^\mu \lambda^J) + \frac{1}{2\sqrt{2}} x_r^M c^{rIJ} (\bar{\chi}^M \gamma^{\mu\nu} \lambda^I) \hat{F}_{\mu\nu}^J \\
& - \sqrt{2} V_\alpha^{aA} \xi^{\alpha I} (\bar{\lambda}_A^I \Psi_a) + \frac{i}{\sqrt{2}} \mathcal{A}_{\alpha B}^A \xi^{\alpha I} (\bar{\lambda}_A^I \gamma^\mu \psi_\mu^B) \\
& + \frac{1}{\sqrt{2}} [(v \cdot c)^{-1} (x^M \cdot c)]^{IJ} \mathcal{A}_{\alpha B}^A \xi^{\alpha I} (\bar{\lambda}_A^J \chi^{MB}) \\
& + \frac{1}{8} (\bar{\chi}^M \gamma^{\mu\nu\rho} \chi^M) (\bar{\psi}_\mu \gamma_\nu \psi_\rho) - \frac{1}{8} (\bar{\chi}^M \gamma^\mu \chi^N) (\bar{\chi}^M \gamma_\mu \chi^N) \\
& + \frac{1}{8} (\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) (\psi_\mu \gamma_\nu \psi_\rho) + \frac{1}{48} \Omega^{abcd} (\bar{\Psi}_a \gamma_\mu \Psi_b) (\bar{\Psi}_c \gamma^\mu \Psi_d) \\
& + \frac{1}{32} v_r c^{rIJ} (\bar{\lambda}^I \gamma_{\mu\nu\rho} \lambda^J) (\bar{\chi}^M \gamma^{\mu\nu\rho} \chi^M) - \frac{i}{16} (\bar{\chi}^M \gamma_{\mu\nu} \psi_\rho) x_r^M c^{rIJ} (\bar{\lambda}^I \gamma^{\mu\nu\rho} \lambda^J) \\
& - \frac{i}{4} x_r^M c^{rIJ} (\bar{\chi}^M \gamma^\mu \gamma^\nu \lambda^I) (\bar{\psi}_\mu \gamma_\nu \lambda^J) + \frac{1}{8} (\bar{\psi}_\mu \gamma_\nu \psi_\rho) v_r c^{rIJ} (\bar{\lambda}^I \gamma^{\mu\nu\rho} \lambda^J) \\
& - \frac{1}{16} v_r c^{rIJ} (\bar{\chi}^M \lambda^I) (\bar{\chi}^M \lambda^J) - \frac{3}{32} v_r c^{rIJ} (\bar{\chi}^M \gamma_{\mu\nu} \lambda^I) (\bar{\chi}^M \gamma^{\mu\nu} \lambda^J) \\
& + [(x^M \cdot c) (v \cdot c)^{-1} (x^N \cdot c)]^{IJ} [-\frac{1}{4} (\bar{\chi}^M \lambda^I) (\bar{\chi}^N \lambda^J) \\
& + \frac{1}{16} (\bar{\chi}^N \gamma^{\mu\nu} \lambda^I) (\bar{\chi}^M \gamma_{\mu\nu} \lambda^J) - \frac{1}{8} (\bar{\chi}^N \lambda^I) (\bar{\chi}^M \lambda^J)] \\
& + \frac{5}{192} v_r c^{rIJ} (\bar{\lambda}^I \gamma_{\mu\nu\rho} \lambda^J) (\bar{\Psi}_a \gamma^{\mu\nu\rho} \Psi^a) - \frac{1}{8} v_r v_s c^{rIJ} c^{sKL} (\bar{\lambda}^I \gamma_\mu \lambda^K) (\bar{\lambda}^J \gamma^\mu \lambda^L) \\
& + \frac{\alpha}{8} c^{rIJ} c_r^{KL} (\bar{\lambda}^I \gamma_\mu \lambda^K) (\bar{\lambda}^J \gamma^\mu \lambda^L) \quad . \tag{3.1}
\end{aligned}$$

The variation of this lagrangian with respect to gauge transformations gives the abelian gauge anomaly

$$\mathcal{A}_\Lambda = -\frac{1}{32} \epsilon^{\mu\nu\rho\sigma\delta\tau} c_r^{IJ} c^{rKL} \Lambda^I F_{\mu\nu}^J F_{\rho\sigma}^K F_{\delta\tau}^L \quad , \tag{3.2}$$

while its variation with respect to the supersymmetry transformations

$$\begin{aligned}
\delta e_\mu^m &= -i(\bar{\epsilon} \gamma^m \psi_\mu) \quad , \\
\delta B_{\mu\nu}^r &= i v^r (\bar{\psi}_{[\mu} \gamma_{\nu]} \epsilon) + \frac{1}{2} x^{Mr} (\bar{\chi}^M \gamma_{\mu\nu} \epsilon) + 2 c^{rIJ} A_{[\mu}^I \delta A_{\nu]}^J \quad , \\
\delta v_r &= x_r^M (\bar{\epsilon} \chi^M) \quad , \quad \delta x_r^M = v_r (\bar{\epsilon} \chi^M) \quad , \\
\delta \phi^\alpha &= V_{aA}^\alpha (\bar{\epsilon}^A \Psi^a) \quad , \\
\delta A_\mu^I &= -\frac{i}{\sqrt{2}} (\bar{\epsilon} \gamma_\mu \lambda^I) \quad ,
\end{aligned}$$

$$\begin{aligned}
\delta\psi_\mu^A &= D_\mu(\hat{\omega})\epsilon^A + \frac{1}{4}v_r\hat{H}_{\mu\nu\rho}^r\gamma^{\nu\rho}\epsilon^A - \frac{3i}{8}\gamma_\mu\chi^{MA}(\bar{\epsilon}\chi^M) - \frac{i}{8}\gamma^\nu\chi^{MA}(\bar{\epsilon}\gamma_{\mu\nu}\chi^M) \\
&\quad + \frac{i}{16}\gamma_{\mu\nu\rho}\chi^{MA}(\bar{\epsilon}\gamma^{\nu\rho}\chi^M) - \frac{9i}{16}v_rc^{rIJ}\lambda^{IA}(\bar{\epsilon}\gamma_\mu\lambda^J) + \frac{i}{16}v_rc^{rIJ}\gamma_{\mu\nu}\lambda^{IA}(\bar{\epsilon}\gamma^\nu\lambda^J) \\
&\quad - \frac{i}{32}v_rc^{rIJ}\gamma^{\nu\rho}\lambda^{IA}(\bar{\epsilon}\gamma_{\mu\nu\rho}\lambda^J) - \delta\phi^\alpha\mathcal{A}_{\alpha B}^A\psi_\mu^B \quad , \\
\delta\chi^{MA} &= \frac{i}{2}x_r^M(\partial_\mu\hat{v}^r)\gamma^\mu\epsilon^A + \frac{i}{12}x_r^M\hat{H}_{\mu\nu\rho}^r\gamma^{\mu\nu\rho}\epsilon^A \\
&\quad + \frac{1}{4}x_r^M c^{rIJ}\gamma_\mu\lambda^{IA}(\bar{\epsilon}\gamma^\mu\lambda^J) - \delta\phi^\alpha\mathcal{A}_{\alpha B}^A\chi^{MB} \quad , \\
\delta\Psi^a &= i\gamma^\mu\epsilon_A V_\alpha^{aA}D_\mu\phi^\alpha - \delta\phi^\alpha\mathcal{A}_{\alpha b}^a\Psi^b \quad , \\
\delta\lambda^{IA} &= -\frac{1}{2\sqrt{2}}\hat{F}_{\mu\nu}^I\gamma^{\mu\nu}\epsilon^A + [(v\cdot c)^{-1}(x^M\cdot c)]^{IJ}[-\frac{1}{2}(\bar{\chi}^M\lambda^J)\epsilon^A - \frac{1}{4}(\bar{\chi}^M\epsilon)\lambda^{JA} \\
&\quad + \frac{1}{8}(\bar{\chi}^M\gamma_{\mu\nu}\epsilon)\gamma^{\mu\nu}\lambda^{JA}] - \delta\phi^\alpha\mathcal{A}_{\alpha B}^A\lambda^{IB} - \frac{1}{\sqrt{2}}[(v\cdot c)^{-1}]^{IJ}\mathcal{A}_{\alpha B}^A\xi^{\alpha J}\epsilon^B \quad (3.3)
\end{aligned}$$

gives the supersymmetry anomaly

$$\begin{aligned}
\mathcal{A}_\epsilon &= c_r^{IJ}c^{rKL}\{-\frac{1}{16}\epsilon^{\mu\nu\rho\sigma\delta\tau}\delta_\epsilon A_\mu^I A_\nu^J F_{\rho\sigma}^K F_{\delta\tau}^L - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\delta\tau}\delta_\epsilon A_\mu^I F_{\nu\rho}^J A_\sigma^K F_{\delta\tau}^L \\
&\quad + \frac{ie}{8}\delta_\epsilon A_\mu^I F_{\nu\rho}^J (\bar{\lambda}^K\gamma^{\mu\nu\rho}\lambda^L) + \frac{ie}{8}\delta_\epsilon A_\mu^I (\bar{\lambda}^J\gamma^{\mu\nu\rho}\lambda^K)F_{\nu\rho}^L + \frac{ie}{4}\delta_\epsilon A_\mu^I (\bar{\lambda}^J\gamma_\nu\lambda^K)F^{L\mu\nu} \\
&\quad + \frac{e}{128}\delta_\epsilon e_\mu^m (\bar{\lambda}^I\gamma^{\mu\nu\rho}\lambda^J)(\bar{\lambda}^K\gamma_{m\nu\rho}\lambda^L) - \frac{e}{8\sqrt{2}}\delta_\epsilon A_\mu^I (\bar{\lambda}^J\gamma^\mu\gamma^\nu\gamma^\rho\lambda^K)(\bar{\lambda}^L\gamma_\nu\psi_\rho)\} \\
&\quad + ec_r^{IJ}[c^r(v\cdot c)^{-1}(x^M\cdot c)]^{KL}\{-\frac{i}{4\sqrt{2}}\delta_\epsilon A_\mu^I (\bar{\lambda}^J\gamma^\mu\lambda^K)(\bar{\chi}^M\lambda^L) \\
&\quad + \frac{i}{16\sqrt{2}}\delta_\epsilon A_\mu^I (\bar{\lambda}^J\gamma^\mu\gamma^{\nu\rho}\lambda^L)(\bar{\chi}^M\gamma_{\nu\rho}\lambda^K) - \frac{i}{8\sqrt{2}}\delta_\epsilon A_\mu^I (\bar{\lambda}^J\gamma^\mu\lambda^L)(\bar{\chi}^M\lambda^K)\} \\
&\quad - \frac{ie}{4}c_r^{IJ}[(v\cdot c)^{-1}c^r]^{KL}\delta_\epsilon A_\mu^I \mathcal{A}_{\alpha B}^A \xi^{\alpha K}(\bar{\lambda}_A^L\gamma^\mu\lambda^{JB}) \\
&\quad + \frac{\alpha}{8}c_r^{IJ}c^{rKL}\delta_\epsilon[e(\bar{\lambda}^I\gamma_\mu\lambda^K)(\bar{\lambda}^J\gamma^\mu\lambda^L)] \quad . \quad (3.4)
\end{aligned}$$

Once again, in the case of the gauginos, aside from local symmetry transformations and field equations, the commutator of two supersymmetry transformations generates the additional two-cocycle

$$\begin{aligned}
\delta_{(\alpha)}\lambda^I &= [(v\cdot c)^{-1}c_r]^{IJ}c^{rKL}[-\frac{1}{8}(\bar{\epsilon}_1\gamma_\mu\lambda^K)(\bar{\epsilon}_2\gamma_\nu\lambda^L)\gamma^{\mu\nu}\lambda^J - \frac{\alpha}{4}(\bar{\lambda}^J\gamma_\mu\lambda^K)(\bar{\epsilon}_1\gamma_\nu\lambda^L)\gamma^{\mu\nu}\epsilon_2 \\
&\quad + \frac{\alpha}{32}(\bar{\lambda}^J\gamma_{\mu\nu\rho}\lambda^K)(\bar{\epsilon}_1\gamma^\rho\lambda^L)\gamma^{\mu\nu}\epsilon_2 + \frac{\alpha}{32}(\bar{\lambda}^J\gamma_\rho\lambda^K)(\bar{\epsilon}_1\gamma^{\mu\nu\rho}\lambda^L)\gamma_{\mu\nu}\epsilon_2 \\
&\quad + \frac{1-\alpha}{8}(\bar{\lambda}^J\gamma_\mu\lambda^K)(\bar{\epsilon}_1\gamma^\mu\lambda^L)\epsilon_2 - (1\leftrightarrow 2) \\
&\quad + \frac{1-\alpha}{32}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)(\bar{\lambda}^K\gamma_{\mu\nu\rho}\lambda^L)\gamma^{\nu\rho}\lambda^J] \quad . \quad (3.5)
\end{aligned}$$

All the observations made for the non-abelian case are naturally valid also here: the

theory is obtained by the requirement that the Wess-Zumino conditions close on-shell, and, as we have already shown, it is determined up to an arbitrary quartic coupling for the gauginos. In the case of a single vector multiplet, in which this quartic coupling vanishes, the two-cocycle of eq. (3.5) is still present, although it is properly independent of α . The tensionless string phase transition point in the moduli space of the scalars in the tensor multiplets now would correspond to the vanishing of some of the eigenvalues of the matrix $(v \cdot c)^{IJ}$ [11].

4 Discussion

In the previous sections we have completed the coupling of (1,0) six-dimensional supergravity to tensor multiplets, vector multiplets and charged hypermultiplets. We have derived all the field equations from the lagrangian (2.38), with the prescription that the (anti)self-duality conditions of eq. (2.28) must be used after varying. Moreover, the variation of the lagrangian with respect to the 2-forms gives the divergence of the (anti)self-duality conditions. We want now to apply to our case the general method introduced by Pasti, Sorokin and Tonin [14] for obtaining Lorentz-covariant lagrangians for (anti)self-dual tensors using a single auxiliary field. Alternative constructions [19], some of which preceded the work of PST, need an infinite number of auxiliary fields, and bear a closer relationship to the BRST formulation of closed-string spectra [20]. This method has already been applied to a number of systems, including (1,0) six-dimensional supergravity coupled to tensor multiplets [21], type IIB ten-dimensional supergravity [22] and (1,0) six-dimensional supergravity coupled to vector and tensor multiplets [23, 11].

Our theory describes a single self-dual 3-form

$$\hat{\mathcal{H}}_{\mu\nu\rho} = v_r \hat{H}_{\mu\nu\rho}^r - \frac{i}{8} (\bar{\chi}^M \gamma_{\mu\nu\rho} \chi^M) + \frac{i}{8} (\bar{\Psi}_a \gamma_{\mu\nu\rho} \Psi^a) \quad (4.1)$$

and n_T antiself-dual 3-forms

$$\hat{\mathcal{H}}_{\mu\nu\rho}^M = x_r^M \hat{H}_{\mu\nu\rho}^r - \frac{i}{4} x_r^M c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) \quad . \quad (4.2)$$

The complete Lagrangian is obtained adding to eq. (2.38) the term

$$-\frac{\partial^\mu\phi\partial^\sigma\phi}{4(\partial\phi)^2}[\hat{\mathcal{H}}_{\mu\nu\rho}^-\hat{\mathcal{H}}_\sigma^{-\nu\rho}+\hat{\mathcal{H}}_{\mu\nu\rho}^{M+}\hat{\mathcal{H}}_\sigma^{M+\nu\rho}] \quad , \quad (4.3)$$

where ϕ is an auxiliary field and $H^\pm = H \pm *H$. The resulting lagrangian is invariant under the additional gauge transformations [14]

$$\delta B_{\mu\nu}^r = (\partial_\mu\phi)\Lambda_\nu^r - (\partial_\nu\phi)\Lambda_\mu^r \quad (4.4)$$

and

$$\delta\phi = \Lambda \quad , \quad \delta B_{\mu\nu}^r = \frac{\Lambda}{(\partial\phi)^2}[v^r\hat{\mathcal{H}}_{\mu\nu\rho}^- - x^{Mr}\hat{\mathcal{H}}_{\mu\nu\rho}^{M+}]\partial^\rho\phi \quad , \quad (4.5)$$

used to recover the usual field equations for (anti)self-dual forms. The 3-form

$$\hat{K}_{\mu\nu\rho} = \hat{\mathcal{H}}_{\mu\nu\rho} - 3\frac{\partial_{[\mu}\phi\partial^\sigma\phi}{(\partial\phi)^2}\hat{\mathcal{H}}_{\nu\rho]\sigma}^- \quad (4.6)$$

is *identically* self-dual, while the 3-forms

$$\hat{K}_{\mu\nu\rho}^M = \hat{\mathcal{H}}_{\mu\nu\rho}^M - 3\frac{\partial_{[\mu}\phi\partial^\sigma\phi}{(\partial\phi)^2}\hat{\mathcal{H}}_{\nu\rho]\sigma}^{M+} \quad (4.7)$$

are *identically* antiself-dual [21]. In order to obtain the complete supersymmetry transformations, we have to substitute $\hat{\mathcal{H}}$ with \hat{K} in the transformation of the gravitino and $\hat{\mathcal{H}}^M$ with \hat{K}^M in the transformations of the tensorinos. Moreover, the auxiliary scalar is invariant under supersymmetry [21, 22]. It can be shown that the complete lagrangian transforms under supersymmetry as dictated by the Wess-Zumino consistency conditions. The commutator of two supersymmetry transformations on $B_{\mu\nu}^r$ now generates the local PST transformations with parameters

$$\Lambda_{r\mu} = \frac{\partial^\sigma\phi}{(\partial\phi)^2}(v_r\hat{\mathcal{H}}_{\sigma\mu\nu}^- - x_r^M\hat{\mathcal{H}}_{\sigma\mu\nu}^{M+})\xi^\nu \quad , \quad \Lambda = \xi^\mu\partial_\mu\phi \quad , \quad (4.8)$$

while in the parameter of the local Lorentz transformation the term $\hat{\mathcal{H}}$ is replaced by \hat{K} . All other parameters remain unchanged.

It would be interesting to study in some detail the vacua of the lagrangian (2.38), analyzing the extrema of the potential (2.23). As a simple example, consider the model without hypermultiplets, in which one can gauge the global R-symmetry group $USp(2)$

of the theory. Formally, the gauged theory without hypermultiplets is obtained from the theory described previously putting $n_H = 0$ and making the identification

$$\mathcal{A}_{\alpha B}^A \xi^{\alpha i} \rightarrow -T^{iA}{}_B \quad , \quad (4.9)$$

where T^i are the anti-hermitian generators of $USp(2)$. This corresponds to the replacement of the previous couplings between gauge fields and spinors, dressed by the scalars in the case $n_H \neq 0$, with ordinary minimal couplings:

$$D_\mu \phi^\alpha \mathcal{A}_{\alpha B}^A \epsilon^B \rightarrow A_\mu^A{}_B \epsilon^B \quad . \quad (4.10)$$

Implementing this identification gives in this case the positive-definite potential

$$V = \frac{3}{8v_r c^{r1}} \quad (4.11)$$

for the scalars in the tensor multiplets. One would thus expect that in these models supersymmetry be spontaneously broken. Notice that this potential diverges at the tensionless string phase transition point. Similarly, one could try to study explicitly the behavior of the potential in simple models containing charged hypermultiplets. Their dimensional reduction gives N=2 supergravity coupled to vector and hypermultiplets in five dimensions, and in the context of the AdS/CFT correspondence and its generalizations [24] there is a renewed interest in studying the explicit gauging of these five-dimensional models (see, for instance, [25] and references therein). Notice that in five dimensions the anomaly that results from the dimensional reduction of our model can be canceled by a local counterterm, and thus the low-energy effective action does not present the subtleties of the six-dimensional case [6].

The couplings we have derived here are the most general couplings of (1,0) six-dimensional supergravity to vector, tensor and hypermultiplets. One may wonder if one had the option to gauge a subgroup of $SO(1, n_T)$, the isometry group of the scalars in the tensor multiplets. Of course, we do not know how to write a gauge covariant field-strength for antisymmetric tensor fields, but there is a more direct reason why this gauging is not expected to work, namely the fact that once we couple vector and tensor multiplets, the $SO(1, n_T)$ transformations are no longer global symmetries of the theory, because of the presence of the matrices c^r .

Acknowledgments

I am grateful to C. Angelantonj, S. Ferrara and A. Sagnotti for interesting and helpful discussions. I also thank the organizers of the “Semestre on Supergravity, Superstrings and M-Theory” and all the members of the Centre Emile Borel for the hospitality during the period in which this work has been completed. This work has been supported in part by a Marie Curie Fellowship of the European Community Programme “Improving human research potential and the socio-economic knowledge base” under contract number HPMT-CT-2000-00165, in part by the EEC contract HPRN-CT-2000-00122 and in part by the INTAS project 991590.

Appendix

The conventions used in this paper are similar to those of [7]. The indices of $USp(2)$ and $USp(2n_H)$ are raised and lowered by the antisymmetric symplectic invariant tensors ϵ^{AB} and Ω^{ab} with the following conventions:

$$V^A = \epsilon^{AB} V_B \quad , \quad V_A = \epsilon_{BA} V^B \quad (\epsilon^{AB} \epsilon_{AC} = \delta_C^B) \quad ,$$

$$W^a = \Omega^{ab} W_b \quad , \quad W_a = \Omega_{ba} W^b \quad (\Omega^{ab} \Omega_{ac} = \delta_c^b) \quad .$$

All spinors satisfy symplectic Majorana-Weyl conditions. In particular, spinors with $USp(2)$ indices satisfy the condition

$$\Psi^A = \epsilon^{AB} C \bar{\Psi}_B^T \quad ,$$

while spinors with $USp(2n_H)$ indices satisfy the condition

$$\Psi^a = \Omega^{ab} C \bar{\Psi}_b^T \quad ,$$

where

$$\bar{\Psi}_{A,a} = (\Psi^{a,A})^\dagger \gamma_0 \quad .$$

From these relations one can deduce the properties of spinor bilinears under Majorana flip. For instance:

$$(\bar{\chi}_A \Psi^a) = \epsilon_{AB} \Omega^{ab} (\bar{\Psi}_b \chi^B) \quad ,$$

and similar relations when γ -matrices are included. In our notations a spinor bilinear with two $USp(2)$ indices contracted is written without explicit indices, *i.e.*

$$(\bar{\chi}_A \Psi^A) \equiv (\bar{\chi} \Psi) \quad ,$$

while in all the other bilinears the symplectic indices are explicit.

The connections $\mathcal{A}_{\alpha B}^A$ and $\mathcal{A}_{\alpha b}^a$ are anti-hermitian. Belonging to the adjoint representation of a symplectic group, they are symmetric if considered with both upper or both lower indices.

The anti-hermitian generators T^i and T^I satisfy the commutation relations

$$[T^i, T^j] = \epsilon^{ijk} T^k \quad , \quad [T^I, T^J] = f^{IJK} T^K \quad ,$$

as well as the trace conditions

$$tr(T^i T^j) = -\frac{1}{2} \delta^{ij} \quad , \quad tr(T^I T^J) = -\frac{1}{2} \delta^{IJ} \quad .$$

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